

Gap Solitons in Almost Periodic One-Dimensional Structures

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Abstract

We consider almost periodic stationary nonlinear Schrödinger equations in dimension 1. Under certain assumptions we prove the existence of nontrivial finite energy solutions in the strongly indefinite case. The proof is based on a careful analysis of the energy functional restricted to the so-called generalized Nehari manifold, and the existence and fine properties of special Palais-Smale sequences. As an application, we show that certain one dimensional almost periodic photonic crystals possess gap solitons for all prohibited frequencies.

Keywords: Nonlinear Schrödinger equation, variational methods, strongly indefinite functional, almost periodicity, generalized Nehari manifold, finite energy solution.

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1 Introduction

In this paper we consider the problem of existence of non-zero finite energy solutions (also known as bound states, or homoclinics) to the following one-dimensional stationary nonlinear Schrödinger equation (NLS)

$$-u'' + V(x)u = f(x, u)$$

in which the x -dependence is almost periodic, while the linear part of the equation is not necessarily positive definite. More precisely, we suppose that 0 is not in the spectrum of the linear part. The most interesting case is when 0 belongs to a finite spectral gap, *i.e.*, there is a non-empty part of the spectrum below zero. It is well-known that the spectrum of a periodic Schrödinger operator is absolutely continuous and has the so-called band-gap structure. Moreover, typical one-dimensional Schrödinger operators have infinitely many gaps [23].

In the almost periodic case the spectrum is not absolutely continuous in general. However, it possesses gaps. Moreover, typically the spectrum of an almost periodic Schrödinger operator is nowhere dense (see, *e.g.*, [3, 17]).

In last decades, the periodic NLS in arbitrary dimension has been studied extensively including strictly indefinite case (see, *e.g.*, [6, 10, 14, 16, 18, 29, 31] and references therein). In the almost periodic case the situation is totally different. The first result in this direction obtained by variational methods concerns second order Hamiltonian systems with positive definite linear part [24], including one-dimensional NLS equation. This result has been developed in several directions, but still for problems with positive definite linear part (see [2, 5, 13, 19, 20, 27, 30]).

One of the key ingredients in [24] is the construction of special Palais-Smale sequences, known as (\overline{PS}) sequences [4], based on mountain pass geometry and (negative) gradient flow of the associated functional J . Notice that the mountain pass minimax class is invariant with respect to standard deformations and, hence, the gradient flow. In the strictly indefinite case the functional J possesses infinite dimensional linking geometry [10, 31]. However, the minimax class related to this geometry is not invariant with respect to the gradient flow. We overcome this difficulty by employing the generalized Nehari manifold of the functional J in its original version introduced in [16]. Special Palais-Smale sequences are then constructed directly via the negative gradient flow of the functional J restricted to the generalized Nehari manifold. Notice that this requires certain additional smoothness of the nonlinearity with respect to u to guarantee the existence and uniqueness for such flow.

An essential part of [24] is devoted to detailed structure of Palais-Smale sequences with the aim to relate special Palais-Smale sequences and returning sequences of real numbers for the functional J . The arguments are quite involved and depend crucially on the positivity of the linear part. In our work we restrict ourselves to Palais-Smale sequences at levels close to the ground level, which is the infimum of J over the Nehari manifold. The structure of such sequences is not complicated so that to pass to a returnig sequence it is enough to use relatively simple concentration-compactness arguments.

In addition, let us point out that in this paper we use a weaker concept of almost periodicity, the so-called Stepanov almost periodicity. We do that to allow piece-wise continuous dependence of the potential $V(x)$ and nonlinearity $f(x, u)$ on x . This is important in the application of our result to nonlinear optics.

Now let us turn to applications. The term *gap soliton* was born in the area of photonic crystals. Photonic crystals are optical media with spatially periodic, or close-to-periodic, structure. Here close-to-periodic can be almost periodic, or asymptotically periodic, or something similar. In this context almost periodicity models disordered periodic structures, while asymptotic periodicity represents a localized defect in a periodic structure. One of the basic fiture of photonic crystals is that light of certain frequencies (so-called prohibited frequencies) can not propagate through such a medium. This is due to the band-gap

structure of the spectrum of, say, periodic Maxwell operators. Actually, prohibited frequencies are exactly the points in gaps of spectrum. However, if a photonic crystal is made of non-linear media, a completely new phenomenon occur. In such crystals there may exist localized light pattern with prohibited carrier frequencies. These are called gap solitons. For physics and mathematics of photonic crystals we refer to [1, 8, 9, 12, 16, 26] and references therein.

Gap solitons are widely studied in physics literature by means of numerical and asymptotical methods. However, to the best of our knowledge there is only one rigorous mathematical result on the existence of gap solitons [16]. It concerns gap solitons of special form (the so-called *TM*-mode) in one- and two-dimensional periodic crystals. As we will see below, our main result provides the existence of gap solitons in one-dimensional almost periodic photonic crystals.

Notice that the stationary NLS also appears as an equation for the profile function of a standing wave in the evolutionally nonlinear Schrödinger equation. Typically, such standing waves exist if its frequency belongs to a spectral gap of the linear part. Often such waves are also called gap solitons. Certainly, our result provides the existence of such waves under appropriate assumptions.

The organization of the paper is as follows. Section 2 contains certain facts on one-dimensional Schrödinger operators and reminds the concept of Stepanov almost periodicity. In Section 3 we formulate our main result, while Section 4 is devoted to a variational formulation of the problem and certain simple results on the continuous dependence of the energy functional on the envelope of the problem. Sections 5 and 6 form a core of our techniques, and are devoted to the generalized Nehari manifold and Palais-Smale sequences, respectively. The proof of main result is contained in Section 7. Finally, in Section 8, we sketch an application to photonic crystals.

2 Preliminaries

First, let us introduce basic spaces of real valued functions on \mathbb{R} .

By $L^2(\mathbb{R})$ we denote the space of square integrable functions endowed with the standard norm $\|\cdot\|_2$ and inner product (\cdot, \cdot) . The Sobolev space

$$H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid u' \in L^2(\mathbb{R})\}$$

with the graph norm $\|\cdot\|$ is a Hilbert space. The inner product in $H^1(\mathbb{R})$ is denoted by $\langle \cdot, \cdot \rangle$. By $L^\infty(\mathbb{R})$ we denote the space of all essentially bounded functions with the standard norm $\|\cdot\|_\infty$. The space of all infinitely differentiable compactly supported functions is denoted by $C_0^\infty(\mathbb{R})$.

By $H^{-1}(\mathbb{R})$ we denote the dual space to $H^1(\mathbb{R})$ with the norm $\|\cdot\|_*$. The symbol (\cdot, \cdot) stands both for the inner product in $L^2(\mathbb{R})$ and for the duality pairing on $H^{-1}(\mathbb{R}) \times H^1(\mathbb{R})$. This does not lead to any confusion. It is well-known that

$$H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \subset H^{-1}(\mathbb{R})$$

continuously and densely. Moreover, $H^1(\mathbb{R})$ is continuously embedded into $L^\infty(\mathbb{R})$. Actually, any H^1 -function is continuous and vanishes at infinity.

A locally integrable function u is *Stepanov bounded* if

$$\|u\|_{BS} = \sup_{t \in \mathbb{R}} \int_t^{t+1} |u(x)| dx < \infty.$$

Such functions form a Banach space denoted by $BS(\mathbb{R})$. A function $u \in BS(\mathbb{R})$ is *Stepanov almost periodic* if the set of its shifts

$$\{T_z u\}_{z \in \mathbb{R}},$$

where $(T_z)u(x) = u(x+z)$, is precompact in the space $BS(\mathbb{R})$. In other words, for any sequence $z_k \in \mathbb{R}$ there exists a subsequence $z_{k'}$ such that the sequence $T_{z_{k'}} u$ converges in the space $BS(\mathbb{R})$. The space of Stepanov almost periodic functions is a closed subspace of $BS(\mathbb{R})$ denoted by $S(\mathbb{R})$. For a Stepanov almost periodic function u , the closure of $\{T_z u\}_{z \in \mathbb{R}}$ in the space $BS(\mathbb{R})$ is denoted by $\mathcal{E}(u)$ and is called the *envelop* of u . The following simple fact is well-known (see, e.g., [11, 15]). If $u_h = \lim T_{z_k} u \in \mathcal{E}(u)$, then $u = \lim T_{-z_k} u_h$ (limits in the space $BS(\mathbb{R})$). The set $\mathcal{E}(u)$ is a compact set in $BS(\mathbb{R})$. Notice that the operators T_z form a strongly continuous group of operators in $S(\mathbb{R})$, but this is not so in the whole space $BS(\mathbb{R})$.

Let $V \in BS(\mathbb{R})$. Then the operator

$$L = L_0 + V(x) = -\frac{d^2}{dx^2} + V(x), \quad (2.1)$$

defined by means of the sum of quadratic forms associated to L_0 and V , is a bounded below self-adjoint operator in $L^2(\mathbb{R})$. The form domain of L is the space $H^1(\mathbb{R})$. Furthermore, the operator L extends to a bounded linear operator from $H^1(\mathbb{R})$ into $H^{-1}(\mathbb{R})$ still denoted by L , and the extension depends continuously on $V \in BS(\mathbb{R})$ with respect to the operator norm, hence, with respect to the norm resolvent convergence (see, e.g., [25]). Furthermore, the operator of multiplication by V is a bounded linear operator from $H^1(\mathbb{R})$ into $H^{-1}(\mathbb{R})$ and its norm does not exceed $\|V\|_{BS}$, i.e.,

$$|(Vu, v)| = \left| \int_{\mathbb{R}} V(x)u(x)v(x)dx \right| \leq \|V\|_{BS}\|u\|\|v\|. \quad (2.2)$$

Moreover, this operator is L_0 -form bounded with form bound 0 [25].

In what follows we denote by $\sigma(L)$ the spectrum of L . If $0 \notin \sigma(L)$, we denote by $E^+ \subset H^1(\mathbb{R})$ and $E^- \subset H^1(\mathbb{R})$ the positive and negative subspaces of the form (Lu, u) , respectively. These subspaces are orthogonal with respect to both H^1 and L^2 inner products. Moreover, LE^\pm is orthogonal to E^\mp with respect to duality pairing (\cdot, \cdot) on $H^{-1}(\mathbb{R}) \times H^1(\mathbb{R})$. By P^+ and P^- we denote the orthogonal projectors in $H^1(\mathbb{R})$ onto E^+ and E^- , respectively. Notice that these projectors are orthogonal with respect to (\cdot, \cdot) as well. Each element $u \in H^1$ possesses the representation $u = u^+ + u^-$, where $u^+ = P^+u$ and $u^- = P^-u$.

Proposition 2.1 *Let $V \in BS(\mathbb{R})$. If $0 \notin \sigma(L)$, then there exists a constant $\kappa > 0$, depending on $\|V\|_{BS}$ and the distance between 0 and $\sigma(L)$, such that*

$$(Lu, u) \geq \kappa \|u\|^2, \quad u \in E^+, \quad (2.3)$$

and

$$(Lu, u) \leq -\kappa \|u\|^2, \quad u \in E^-, \quad (2.4)$$

Proof. We prove inequality (2.3), the other is similar.

Let 2δ be the distance between zero and $\sigma(L)$. Then

$$(Lu, u) \geq 2\delta \|u\|_2^2, \quad u \in E^+.$$

Since V is L_0 -form bounded with form bound 0, then there exist sufficiently small $\alpha \in (0, 1)$ and $\beta > 0$, depending on $\|V\|_{BS}$, such that

$$|(Vu, u)| \leq \alpha(L_0u, u) + \beta \|u\|_2^2 = \alpha \|u\|_2^2 + \beta \|u\|_2^2, \quad u \in H^1(\mathbb{R}).$$

Hence, for all $u \in E^+$,

$$(1 - \alpha) \|u\|^2 \leq (Lu, u) + C \|u\|_2^2,$$

where $C = 1 + \beta - \alpha$. The right hand side of this inequality can be expressed as

$$\frac{C + \delta}{\delta} \left[\frac{\delta}{C + \delta} ((Lu, u) - \delta \|u\|_2^2) + \delta \|u\|_2^2 \right].$$

Since $\delta/(C + \delta) < 1$, on the subspace E^+ this quantity does not exceed

$$\frac{C + \delta}{\delta} (Lu, u),$$

and the result follows. \square

Now suppose that $V \in S(\mathbb{R})$. The *envelop* $\mathcal{E}(L)$ of L is the set of all operators L_h of the form (2.1) generated by potentials $V_h \in \mathcal{E}(V)$. Being considered as a subset in the Banach space of all bounded linear operators from $H^1(\mathbb{R})$ into $H^{-1}(\mathbb{R})$, the envelop $\mathcal{E}(L)$ is a compact set.

Proposition 2.2 *Suppose that $V \in S(\mathbb{R})$. Then $\sigma(L_h) = \sigma(L)$ for all $L_h \in \mathcal{E}(L)$.*

Proof. If $V_h \in \mathcal{E}(V)$, then there exists a sequence $z_k \in \mathbb{R}$ such that $T_{z_k} V \rightarrow V_h$ in $BS(\mathbb{R})$. It is easily seen that $\sigma(L_0 + T_{z_k} V) = \sigma(L)$. Since $L_0 + T_{z_k} V \rightarrow L_h$ with respect to the norm resolvent convergence, then, by [21, Theorem VIII.23], $\sigma(L) \subset \sigma(L_h)$. But $T_{-z_k} V_h \rightarrow V$ in $BS(\mathbb{R})$. Hence, interchanging the role of V and V_h , we obtain the required. \square

Remark 2.1 If $V \in S(\mathbb{R})$ and $0 \notin \sigma(L) = \sigma(L_h)$, we denote by E_h^+ and E_h^- the positive and negative subspaces of the quadratic form $(L_h u, u)$. By Proposition 2.2, the conclusion of Proposition 2.1 holds for L_h with the same constant κ . Furthermore, positive and negative spectral projectors depend continuously on the potential. More precisely, let $T_{z_k} V \rightarrow V_h$ in $BS(\mathbb{R})$, and let P_k^\pm and P_h^\pm be the positive (negative) spectral projector that correspond to the potentials $T_{z_k} V$ and V_h , respectively. Then $P_k^\pm \rightarrow P_h^\pm$ with respect to the operator norm.

For functions of two variables, $g(x, u)$, we need an appropriate concept of almost periodicity with respect to the first variable $x \in \mathbb{R}$. It is always assumed that such a function is a Carathéodory function, *i.e.*, $g(x, u)$ is continuous in u for almost all $x \in \mathbb{R}$, and Lebesgue measurable in x for all $u \in \mathbb{R}$. For any $R > 0$, we set

$$\|g\|_R = \left\| \sup_{|u| \leq R} |g(\cdot, u)| \right\|_{BS}.$$

We say that $g(x, u)$ is *strictly Stepanov almost periodic* in x (in symbols $g \in S(\mathbb{R} \times \mathbb{R})$) if $\|g\|_R < \infty$ for all $R > 0$, and for any sequence $z_k \in \mathbb{R}$ there exist a subsequence $z_{k'}$ and a function g_h such that $\|g_h\|_R < \infty$ for all $R > 0$ and

$$\|T_{z_{k'}} g(\cdot, u) - g_h(\cdot, u)\|_R \rightarrow 0 \quad \forall R > 0.$$

In other words, being considered as a function of $x \in \mathbb{R}$ with values in the (Frechét) space of continuous functions of $u \in \mathbb{R}$, g is a Stepanov almost periodic function. The envelope $\mathcal{E}(g)$ of g consists of all such limit functions g_h . Notice, that any strictly Stepanov almost periodic function is Stepanov almost periodic in x uniformly with respect to $u \in [-R, R] \quad \forall R > 0$, but not vice versa.

3 Statement of Problem and Main Result

We are looking for nonzero vanishing at infinity solutions to the following one-dimensional nonlinear Schrödinger equation

$$-u''(x) + V(x)u(x) = \chi f(x, u(x)), \quad (3.1)$$

where $\chi = \pm 1$.

Let

$$F(x, u) = \int_0^u f(x, s) ds.$$

Throughout the remaining part of the paper we suppose that the following assumptions hold true.

- (i) The potential V is Stepanov almost periodic, $V \in S$, and the spectrum of the operator L does not contain zero. In the case when $\chi = -1$ we suppose in addition that there is a non-empty part of the spectrum below 0.

- (ii) For almost all $x \in \mathbb{R}$, the function $f(x, u)$ is continuously differentiable with respect to $u \in \mathbb{R}$. The functions $F(x, u)$, $f(x, u)$ and $f_u(x, u)$ are strictly Stepanov almost periodic. For any $u \neq 0$, the function $F(x, u)$ is bounded below by a positive constant.
- (iii) The nonlinearity satisfies $f(\cdot, 0) = 0$ and $f_u(\cdot, 0) = 0$. Furthermore, for every $R > 0$ there exists a constant $\mu(R) > 0$ such that

$$|f_u(x, u) - f_u(x, v)| \leq \mu(R)|u - v|, \quad |u|, |v| \leq R.$$

for almost all $x \in \mathbb{R}$.

- (iv) There exists a constant $\theta \in (0, 1)$ such that for almost all $x \in \mathbb{R}$

$$0 < f(x, u)u \leq \theta \cdot f_u(x, u)u^2, \quad u \neq 0.$$

Without loss of generality we suppose that $\mu(R_1) \leq \mu(R_2)$ whenever $R_1 \leq R_2$.

Assumption (i) guarantees that the self-adjoint operator L is well-defined (see Section 2). By the mean value theorem, Assumption (iii) implies that for almost all $x \in \mathbb{R}$

$$|f(x, u)| \leq \mu(R)|u|^2 \tag{3.2}$$

and

$$|F(x, u)| \leq \mu(R)|u|^3 \tag{3.3}$$

whenever $|u| \leq R$. Assumption (iv) implies easily that

$$0 < qF(x, u) \leq f(x, u)u, \quad u \neq 0, \tag{3.4}$$

where $q = (1 + \theta)/\theta > 2$. This is the standard Ambrosetti-Rabinowitz condition. In particular, from (3.4) it follows that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$F(x, u) \geq -\varepsilon|u|^2 + C_\varepsilon|u|^q. \tag{3.5}$$

Notice that in Assumption (ii) it is enough to assume strict Stepanov almost periodicity for f_u only. Then so is for f and F .

Example. The nonlinearity

$$f(x, u) = \alpha(x)|u|^{p-2}u, \tag{3.6}$$

satisfies Assumptions (ii)–(iv) provided $\alpha \in S(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\text{ess inf } \alpha > 0$, and $p \geq 3$.

Under Assumptions imposed above, the set of shifts $\{(T_z V, T_z f)\}_{z \in \mathbb{R}}$ is pre-compact with respect to the topology generated by semi-norms $\|V\|_{BS} + \|f_u\|_R$, $R > 0$. Its closure is denoted by \mathcal{E} . This is a compact set. In what follows we always suppose that the set \mathcal{E} is parameterized, not necessarily in a one-to-one

way, by elements h of an index set $\mathcal{H} \supset \mathbb{R}$. Together with equation (3.1) we consider the following family of equations

$$-u''(x) + V_h(x)u(x) = \chi f_h(x, u(x)), \quad h \in \mathcal{E}. \quad (3.7)$$

These equations form the *envelop* of equation (3.1), which can be identified with \mathcal{E} . Any equation in the envelop satisfies Assumptions (i)–(iv) with the same $\mu(R)$ and θ .

Our main result is the following.

Theorem 3.1 *Under Assumptions (i)–(v) equation (3.1) has a nonzero solution $u \in H^1(\mathbb{R})$. Moreover, the solution u is continuously differentiable and decays at infinity exponentially fast, i.e., there exist positive constants α and β such that*

$$|u(x)| + |u'(x)| \leq \alpha \exp(-\beta|x|).$$

The solution in Theorem 3.1 is a weak solution, i.e.,

$$\int_{\mathbb{R}} (u'(x)\varphi'(x) + V(x)u(x)\varphi(x)) dx = \chi \int_{\mathbb{R}} f(x, u(x))\varphi(x) dx$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

Remark 3.1 *Theorem 3.1 applies to all equations (3.7) in the envelop of equation (3.1).*

Remark 3.2 *Suppose that zero is below the essential spectrum of L , i.e., L is positive definite. If $\chi = -1$, then it is easily seen that equation (3.1) has only trivial solution in $H^1(\mathbb{R})$. If $\chi = 1$ and $V(x) \geq \alpha_0 > 0$, the existence of non-trivial solution is obtained for a wider class of nonlinearities, including (3.6) with $p > 2$ (see [24]). Actually, in [24] the potential is a constant function, while $f(x, u)$ is Bohr almost periodic in x , but the arguments of that paper extend straightforwardly to the case of non-constant potential and Stepanov almost periodic x -dependence.*

4 Variational Formulation

Associated to equation (3.1), we introduce the functional

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}} (|u'(x)|^2 + V(x)u^2(x)) dx - \chi \int_{\mathbb{R}} F(x, u(x)) dx \\ &= \frac{1}{2} (Lu, u) - \chi \Phi(u). \end{aligned} \quad (4.1)$$

Similarly, we introduce the functional J_h associated to equation (3.7). Its non-quadratic part is denoted by Φ_h . The functionals J_h form the envelop of J .

Under the assumptions imposed above, the functional J is a well-defined $C^{2,1}$ -functional on the space $H^1(\mathbb{R})$. Its first and second derivatives are given by

$$(J'(u), v) = (Lu, v) - \chi \int_{\mathbb{R}} f(x, u(x))v(x) dx, \quad u, v \in H^1(\mathbb{R}), \quad (4.2)$$

and

$$(J''(u)v, w) = (Lv, w) - \chi \int_{\mathbb{R}} f_u(x, u(x))v(x)w(x) dx, \quad u, v, w \in H^1(\mathbb{R}). \quad (4.3)$$

Notice that J' is weakly continuous.

Often it is convenient to use gradients of J instead of derivatives. These are defined by

$$\langle \nabla J(u), v \rangle = (J'(u), v)$$

and

$$\langle \nabla^2 J(u)v, w \rangle = (J''(u)v, w)$$

for all $u, v, w \in H^1(\mathbb{R})$. Then $\nabla J(u) \in H^1(\mathbb{R})$, while $\nabla^2 J(u)$ is a linear bounded operator in $H^1(\mathbb{R})$.

Now we estimate the difference between two functionals of the form J_h and its derivative.

Proposition 4.1 *For any $h_i \in \mathcal{H}$, $i = 1, 2$, and any $R > 0$*

$$|J_{h_1}(u) - J_{h_2}(u)| \leq \frac{1}{2} \|V_{h_1} - V_{h_2}\|_{BS} \|u\|^2 + \|(f_{h_1})_u - (f_{h_2})_u\|_R \|u\|^2$$

and

$$\|J'_{h_1}(u) - J'_{h_2}(u)\|_* \leq \|V_{h_1} - V_{h_2}\|_{BS} \|u\| + \|(f_{h_1})_u - (f_{h_2})_u\|_R \|u\|$$

provided $u \in H^1(\mathbb{R})$ with $\|u\| \leq R$.

Proof. By inequality (2.2), both the difference of the linear parts and its derivative are estimated by the first term in the right hand sides.

By Taylor's formula and inequality (2.2)

$$\begin{aligned} |\Phi_{h_1}(u) - \Phi_{h_2}(u)| &\leq \int_{\mathbb{R}} \int_0^1 |(f_{h_1})_u(x, tu(x)) - (f_{h_1})_u(x, tu(x))| (1-t) u^2(x) dt dx \\ &\leq \int_{\mathbb{R}} \sup_{|u| \leq R} |(f_{h_1})_u(x, u) - (f_{h_1})_u(x, u)| u^2(x) dx \\ &\leq \|(f_{h_1})_u - (f_{h_1})_u\|_R \|u\|^2 \end{aligned}$$

which implies the first estimate of the proposition.

The proof of second inequality is similar. □

Proposition 4.2 *If $u_n \rightarrow u_0$ weakly in $H^1(\mathbb{R})$, then*

$$J_h(u_n - u_0) - J_h(u_n) + J_h(u_0) \rightarrow 0 \quad (4.4)$$

and

$$J'_h(u_n - u_0) - J'_h(u_n) + J'_h(u_0) \rightarrow 0 \quad (4.5)$$

strongly in $H^{-1}(\mathbb{R})$ uniformly with respect to $h \in \mathcal{H}$.

Proof. The integrand of non-quadratic part, Ψ_h , of J_h satisfies inequalities (3.2) and (3.3) uniformly with respect to $h \in \mathcal{H}$. Hence, arguing exactly as in [30, Lemma 2.6] we obtain the result of proposition for Ψ_h instead of J_h . Due to linearity of the operator L_h , this implies (4.5) immediately.

The quadratic part of the left-hand side in (4.4) coincides with

$$(L_h u_0, u_0) - (L_h u_0, u_n) \rightarrow 0,$$

and we obtain (4.4) for every individual $h \in \mathcal{H}$. Since the operators L_h form a compact set of bounded linear operators from $H^1(\mathbb{R})$ into $H^{-1}(\mathbb{R})$, this convergence is uniform with respect to $h \in \mathcal{H}$. \square

The proof of Theorem 3.1 is given in the subsequent sections. Obviously, $u = 0$ is a trivial critical point of the functional J . We shall prove that J possesses a nontrivial critical point. *In the course of the proof we consider equation (3.1) in the case when $\chi = 1$.* The other case is completely similar. We only need to replace the functional J by $-J$ and interchange the role of the subspaces E^+ and E^- introduced in Section 2.

5 Generalized Nehari Manifold

The *generalized Nehari manifold* \mathcal{N} of the functional J consists of all nonzero $u \in H^1(\mathbb{R})$ such that

$$(J'(u), u) = 0$$

and

$$(J'(u), v) = 0, \quad \forall v \in E^-.$$

Equivalently, these equations can be written as $\langle \nabla J(u), u \rangle = 0$ and $P_- \nabla J(u) = 0$, respectively. The generalized Nehari manifold of a functional $J_h \in \mathcal{E}$ is denoted by \mathcal{N}_h .

For any $w \notin E^-$ we set

$$E_w = \{sw + v : s > 0, v \in E^-\}$$

and

$$\bar{E}_w = \{sw + v : s \in \mathbb{R}, v \in E^-\}.$$

By the definition of \mathcal{N} , if u is a critical point of $J|_{E_w}$, then $u \in \mathcal{N}$. As consequence, \mathcal{N} contains all nontrivial critical points of J .

Lemma 5.1 *For every $w \notin E^-$, the functional $J|_{E_w}$ attains its positive global maximum.*

Proof. Without loss of generality, we can suppose that $w \in E^+$ and $\|w\| = 1$. If $s \in (0, 1]$, then, by (3.3),

$$J(sw) \geq \frac{s^2}{2}(Lw, w) - \mu_1 s^3.$$

Hence, $J(sw) > 0$ for $s > 0$ small enough.

On the other hand, by (2.4) and (3.5), for any $sw + v \in E_w$

$$J(sw + v) \leq \frac{1}{2}s^2(Lw, w) - \frac{1}{2}\kappa\|v\|^2 + \varepsilon s^2\|w\|_{L^2}^2 + \varepsilon\|v\|_{L^2}^2 - C_\varepsilon\|sw + v\|_{L^q}^q.$$

Since the norm of a projector in a Banach space is ≥ 1 , we have that

$$\|sw + v\|_{L^q} \geq C\|sw\|_{L^q}.$$

Then

$$J(sw + v) \leq \left(\frac{1}{2}(Lw, w) + \varepsilon\|w\|_{L^2}^2\right)s^2 - \left(\frac{1}{2}\kappa - \varepsilon\right)\|v\|^2 - C'_\varepsilon\|w\|_{L^q}^q s^q.$$

Taking ε small enough, we obtain that $J(sw + v) \rightarrow -\infty$ as $\|sw + v\| \rightarrow \infty$.

Obviously, $J|_{E_w}$ is upper weakly semi-continuous. Hence, it attains its (positive) global maximum. \square

Remark 5.1 *As in [29, Proposition 2.3]), one can show that for every $w \notin E^-$ the intersection $\mathcal{N} \cap E_w$ consists of exactly one point which is a unique maximum point of $J|_{E_w}$. But we do not use this fact.*

It is convenient to introduce the functional

$$I(u) = J(u) - \frac{1}{2}(J'(u), u).$$

Obviously, $J(u) = I(u)$ for all $u \in \mathcal{N}$. By inequality (3.4), $I(u) \geq 0$ for all $u \in H^1(\mathbb{R})$.

Now we prove the following technical result.

Lemma 5.2 *There exists a constant $C > 0$ independent of $u \in H^1(\mathbb{R})$ such that*

$$\|u\|^2 \leq C(|(J'(u), u)| + |(J'(u), u^-)| + \mu(\|u\|_\infty)\|u\|_\infty\|u\|^2) \quad (5.1)$$

and

$$\|u\|^2 \leq C(|(J'(u), u)| + |(J'(u), u^-)| + (I^{1/2}(u) + I(u))\|u\|) \quad (5.2)$$

for all $u \in H^1(\mathbb{R})$.

Proof. The identity

$$(J'(u), u^-) = (Lu^-, u^-) - \int_{\mathbb{R}} f(x, u) u^- dx$$

and Proposition 2.1 imply

$$\kappa \|u^-\|^2 \leq -(J'(u), u^-) - \int_{\mathbb{R}} f(x, u) u^- dx. \quad (5.3)$$

Similarly, the identity

$$(J'(u), u) = (Lu^+, u^+) - \int_{\mathbb{R}} f(x, u) u^+ dx + (J'(u), u^-)$$

implies

$$\kappa \|u^+\|^2 \leq (J'(u), u) - (J'(u), u^-) + \int_{\mathbb{R}} f(x, u) u^+ dx. \quad (5.4)$$

Adding inequalities (5.3) and (5.4), we obtain immediately that

$$\begin{aligned} \|u\|^2 &\leq C(|(J'(u), u)| + |(J'(u), u^-)| + \\ &\quad + \int_{\mathbb{R}} |f(x, u)| |u^+| dx + \int_{\mathbb{R}} |f(x, u)| |u^-| dx). \end{aligned} \quad (5.5)$$

Then, by inequality (3.2),

$$\int_{\mathbb{R}} |f(x, u)| |u^\pm| dx \leq \mu(\|u\|_\infty) \|u\|_\infty \int_{\mathbb{R}} |u| |u^\pm| dx \leq \mu(\|u\|_\infty) \|u\|_\infty \|u\|_2 \|u^\pm\|_2.$$

Hence,

$$\begin{aligned} \|u\|^2 &\leq C(|(J'(u), u)| + |(J'(u), u^-)| + \mu(\|u\|_\infty) \|u\|_\infty \|u\|_2 (\|u^+\|_2 + \|u^-\|_2)) \\ &\leq C(|(J'(u), u)| + |(J'(u), u^-)| + \mu(\|u\|_\infty) \|u\|_\infty \|u\|^2), \end{aligned}$$

which proves (5.1).

Now we prove inequality (5.2). Given $u \in H^1(\mathbb{R})$, let

$$S_1 = \{x \in \mathbb{R} : |u(x)| \leq 1\}$$

and $S_2 = \mathbb{R} \setminus S_1$. We introduce the following integrals

$$I_1 = \int_{S_1} |f(x, u)|^2 dx$$

and

$$I_2 = \int_{S_2} |f(x, u)| dx.$$

By inequality (3.2), $f^2(x, u) \leq \mu(1)f(x, u)u$ on S_1 , while on S_2 we have that $|f(x, u)| \leq f(x, u)u$. Then, by inequality (3.4),

$$I(u) \geq (2^{-1} - q^{-1}) \int_{\mathbb{R}} f(x, u)u dx \geq \nu I_k, \quad k = 1, 2, \quad (5.6)$$

for some $\nu > 0$. Since

$$\begin{aligned} \int_{\mathbb{R}} |f(x, u)| |u^\pm| dx &\leq \left(\int_{S_1} |f(x, u)|^2 dx \right)^{1/2} \left(\int_{S_1} |u^\pm|^2 dx \right)^{1/2} + \\ &\quad + \|u^\pm\|_\infty \int_{S_2} |f(x, u)| dx \\ &\leq (I_1^{1/2} + I_2) \|u^\pm\|, \end{aligned}$$

equations (5.5) and (5.6) yield (5.2). \square

Proposition 5.1 *There exists a constant $\varepsilon_0 > 0$ such that $\|u\| \geq \|u\|_\infty \geq \varepsilon_0$, $J(u) \geq \varepsilon_0$ and*

$$\int_{\mathbb{R}} f(x, u)u dx \geq 2\varepsilon_0$$

for all $u \in \mathcal{N}$.

Proof. The first two statements follow immediately from Lemma 5.2. Since $F \geq 0$, we see that $L(u) \geq 2\varepsilon_0$ on \mathcal{N} . Now the last statement follows from the definition of \mathcal{N} . \square

Remark 5.2 *Obviously, Lemma 5.2 and Proposition 5.1 hold for all functionals J_h in the envelop of J with the same constants C and ε_0 . In particular, for any nontrivial critical point u of J_h we have that $\|u\| \geq \|u\|_\infty \geq \varepsilon_0$ and $J_h(u) \geq \varepsilon_0$.*

Let $\bar{E}^- = \mathbb{R} \oplus E^-$. Elements of this space are denoted by $[\tau, v]$, where $\tau \in \mathbb{R}$ and $v \in E^-$. The inner product in this space is still denoted by $\langle \cdot, \cdot \rangle$. We introduce the operator $G : H^1(\mathbb{R}) \rightarrow \bar{E}^-$ by the formula

$$G(u) = [\langle \nabla J(u), u \rangle, P^- \nabla J(u)], \quad u \in H^1(\mathbb{R}).$$

It is not difficult to verify that the operator G is a $C^{1,1}$ map, and its derivative is given by the formula

$$G'(u)v = [\langle \nabla^2 J(u)v, u \rangle + \langle \nabla J(u), v \rangle, P^- \nabla^2 J(u)v]$$

for all $u, v \in H^1(\mathbb{R})$. Notice that $\mathcal{N} = G^{-1}(0) \setminus \{0\}$.

Lemma 5.3 *Let $u_0 \in H^1(\mathbb{R})$, and let*

$$\gamma_0 = \langle \nabla J(u_0), u_0 \rangle$$

and

$$\gamma = P^- \nabla J(u_0) \in E^-.$$

Then, for all $\tau \in \mathbb{R}$ and $v \in E^-$,

$$\begin{aligned} \langle G'(u_0)(\tau u_0 + v), [\tau, v] \rangle &\leq 2\gamma_0 \tau^2 - \kappa \|v\|^2 + \frac{3}{2} \tau^2 \|\gamma\| \\ &\quad + \frac{3}{2} \|\gamma\| \|v\|^2 - \tau^2 (1 - \theta) \int_{\mathbb{R}} f(x, u_0) u_0 dx, \end{aligned} \quad (5.7)$$

where $\kappa > 0$ and $\theta \in (0, 1)$ are constants from Proposition 2.1 and Assumption (iv), respectively.

Proof. Since, by the assumptions,

$$(Lu_0, u_0) = \gamma_0 + \int_{\mathbb{R}} f(x, u_0) u_0 dx$$

and

$$(Lu_0, v) = \langle \gamma, v \rangle + \int_{\mathbb{R}} f(x, u_0) v dx,$$

a straightforward, but a little bit tedious, calculation yields the identity

$$\begin{aligned} \langle G'(u_0)(\tau u_0 + v), [\tau, v] \rangle &= 2\tau^2 \gamma_0 + (Lv, v) + 3\tau \langle \gamma, v \rangle \\ &\quad - \int_{\mathbb{R}} (H(x) \tau^2 + 2K(x) \tau v + M(x) v^2) dx, \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} H(x) &= f_u(x, u_0) u_0^2 - f(x, u_0) u_0, \\ K(x) &= f_u(x, u_0) - f(x, u_0) \end{aligned}$$

and

$$M(x) = f_u(x, u_0).$$

Obviously,

$$|\tau \langle \gamma, v \rangle| \leq \frac{1}{2} \|\gamma\| (\tau^2 + \|v\|^2)$$

and, by Proposition 2.1,

$$(Lv, v) \leq -\kappa \|v\|^2.$$

Therefore, it is enough to show that

$$H(x) \tau^2 + 2K(x) \tau v(x) + M(x) v^2(x) \geq \tau^2 (1 - \theta) f(x, u_0(x)) u_0(x).$$

Notice that this inequality is trivial for all $x \in \mathbb{R}$ such that $u_0(x) = 0$. Suppose now that $u_0(x) \neq 0$. In this case $M(x) \neq 0$, and

$$\begin{aligned} H \tau^2 + 2K \tau v + M v^2 &= \left(H - \frac{K^2}{M} \right) \tau^2 + \left(\sqrt{M} v + \frac{K}{\sqrt{M}} \right)^2 \\ &\geq \left(H - \frac{K^2}{M} \right) \tau^2. \end{aligned}$$

Simplifying and making use of the inequality

$$f_u(x, u_0) \geq \theta^{-1} f_0(x, u_0) u_0^{-1}$$

which follows from Assumption (iv), we obtain that

$$\begin{aligned} \left(H - \frac{K^2}{M} \right) &= f(x, u_0) u_0 - \frac{f^2(x, u_0)}{f_u(x, u_0)} \\ &\geq (1 - \theta) f(x, u_0) u_0. \end{aligned}$$

This implies the required. □

Lemma 5.4 *Let R be any positive number. Then*

(a) *For any $u_0 \in \mathcal{N}$ such that $\|u_0\| \leq R$, the operator*

$$G'(u_0)|_{\bar{E}_{u_0}} : \bar{E}_{u_0} \rightarrow \bar{E}^-$$

is invertible and the norm of its inverse operator $[G'(u_0)|_{\bar{E}_{u_0}}]^{-1}$ is bounded above by a constant that depends on R only.

(b) *The norms of projectors generated by the splitting*

$$\ker G'(u_0) + \bar{E}_{u_0}, \quad u_0 \in \mathcal{N}, \|u_0\| \leq R,$$

are bounded above by a constant that depends on R only.

(c) *The norm*

$$\|u\|_{u_0} = \|u_1\| + \|u_2\|, \quad u_0 \in \mathcal{N},$$

where $u_1 \in \ker G'(u_0)$ and $u_2 \in \bar{E}_{u_0}$, is equivalent to the standard H^1 -norm uniformly with respect to $u_0 \in \mathcal{N}$ with $\|u_0\| \leq R$.

Proof. (a) By Lemma 5.3, with $\gamma_0 = 0$ and $\gamma = 0$, the composition of the isomorphism $[\tau, v] \rightarrow \tau u_0 + v$ and $G'(u_0)$ is a negative definite, hence, invertible operator in \bar{E}^- . The norm of the inverse of above mentioned isomorphism is bounded above by a constant that depends on R only. This implies the required.

(b) The projector onto \bar{E}_{u_0} is given by $[G'(u_0)|_{\bar{E}_{u_0}}]^{-1} \circ G'(u_0)$. Since the operator $G'(u_0)$ is uniformly bounded while $\|u_0\| \leq R$, the result follows.

(c) This is an immediate consequence of (b). □

Inspecting the standard proofs of the Inverse Function and Implicit Function theorems (see, e.g., [7], Theorems 4.1.1 and 4.2.1), we see that the following complements to those theorems hold true.

Proposition 5.2 *Let $\varphi : X \rightarrow Y$ be a $C^{1,1}$ -map between Banach spaces such that the derivative φ' is bounded and globally Lipschitz continuous.*

(a) *Given $c_0 > 0$, there exist $\rho > 0$ and $C > 0$ with the following property. For every $x_0 \in X$ such that $\varphi(x_0)$ is invertible and $\|\varphi'(x_0)^{-1}\| \leq c_0$, the inverse*

function φ^{-1} is defined on the ρ -neighborhood of $\varphi(x_0)$ and its Lipschitz constant does not exceed C .

(b) Given $c_0 > 0$ and $c_1 > 0$, there exist $\rho > 0$ and $C > 0$ with the following property. Let $X = X_1 + X_2$ be any splitting of X with mutually complementary closed subspaces as components such that

$$\|x_1\| + \|x_2\| \leq c_1 \|x_1 + x_2\|$$

for all $x_i \in X_i$, $i = 1, 2$. If $x_0 = x_{1,0} + x_{2,0}$ is such that $\varphi(x_0) = 0$ and the partial derivative $\varphi'_2(x_0)$ along X_2 satisfies

$$\|\varphi'_2(x_0)^{-1}\| \leq c_0,$$

then there exists a unique $C^{1,1}$ function ψ defined on the ρ -neighborhood of $x_{1,0}$ in X_1 such that

$$\varphi(x_1 + \psi(x_1)) = 0, \quad \psi(x_{1,0}) = x_{2,0},$$

and the Lipschitz constant of ψ' is bounded above by C .

Proposition 5.3 *The set \mathcal{N} is a non-empty closed $C^{1,1}$ -sub-manifold of H^1 with the tangent space $T_{u_0} = \ker G'(u_0)$ at $u_0 \in \mathcal{N}$. Furthermore, given $R > 0$, there exist $\rho > 0$ and $C > 0$ such that for every $u_0 \in \mathcal{N}$, with $\|u_0\| \leq R$, there exists a $C^{1,1}$ -diffeomorphism from the ρ -neighborhood of 0 in T_{u_0} onto a neighborhood of u_0 in \mathcal{N} such that the Lipschitz constant of its derivative does not exceed C .*

Proof. The result follows immediately from Proposition 5.2(b) and Lemma 5.4. \square

Remark 5.3 *Since \mathcal{N} is a $C^{1,1}$ -manifold, its tangent spaces form a fiber bundle of class $C^{0,1}$.*

Lemma 5.5 *Given $c_0 > 0$ and $c_1 > 0$, there exist positive numbers $\alpha = \alpha(c_0, c_1)$, $r = r(c_0, c_1)$ and $C = C(c_0, c_1)$ such that, for any $u_0 \in H^1(\mathbb{R}) \setminus E^+$ satisfying*

$$\int_{\mathbb{R}} f(x, u_0) u_0 dx \geq c_0,$$

$\|u_0\| \leq c_1$ and $\|G(u_0)\| \leq \alpha$, the restriction

$$G_{u_0} = G|_{\bar{E}_{u_0}} : \bar{E}_{u_0} \rightarrow \bar{E}^-$$

has a local inverse $G_{u_0}^{-1}$ defined on the open ball $B(G(u_0), r)$ of radius r centered at $G(u_0)$, and $\|(G_{u_0}^{-1})'(\xi)\| \leq C$ for all $\xi \in B(G(u_0), r)$.

Proof. By Lemma 5.3, given $c_0 > 0$ there exists sufficiently small $\alpha > 0$ such that

$$\int_{\mathbb{R}} f(x, u_0) u_0 dx \geq c_0,$$

and $\|G(u_0)\| \leq \alpha$ imply that the operator $(G_{u_0})'(u_0) : \bar{E}_{u_0} \rightarrow \bar{E}^-$ is invertible and the norm of the inverse operator $[(G_{u_0})'(u_0)]^{-1}$ is bounded by a constant, say, $c_2 > 0$ that depends on c_0 , c_1 and α , hence, on c_0 and c_1 only. As consequence, there exists a local inverse map $G_{u_0}^{-1}$ in a neighborhood of $\xi_0 = G(u_0)$.

Now the result follows from Proposition 5.2(a). \square

Let us introduce the following quantities

$$m = \inf\{J(u) : u \in \mathcal{N}\}$$

and

$$m_h = \inf\{J_h(u) : u \in \mathcal{N}_h\} \quad h \in \mathcal{H}.$$

By Proposition 5.1 and Remark 5.2, these numbers are strictly positive.

Proposition 5.4 *For all functionals in the envelop of J we have that $m_h = m$.*

Proof. If $h \in \mathcal{H}$, then there exists a sequence $h_k \in \mathbb{R}$ such that

$$V_{h_k} = V(\cdot + h_k)$$

converges to V_h , while

$$F_{h_k} = F(\cdot + h_k, \cdot),$$

$$f_{h_k} = f(\cdot + h_k, \cdot)$$

and

$$(f_{h_k})_u = f_u(\cdot + h_k, \cdot)$$

converge to F_h , f_h and $(f_h)_u$, respectively, in the sense described in Section 2.

Let $\varepsilon > 0$, and let $u \in \mathcal{N}_h$ be such that

$$J_h(u) \leq m_h + \varepsilon.$$

Setting $u_k = u(\cdot - h_k)$, it is easily seen that $\|u_k\| = \|u\|$. By Proposition 4.1,

$$\int_{\mathbb{R}} f(x, u_k) u_k dx = \int_{\mathbb{R}} f(x + h_k, u) u dx \rightarrow \int_{\mathbb{R}} f_h(x, u) u dx,$$

and

$$J(u_k) = J_{h_k}(u) \rightarrow J_h(u).$$

In addition, making use of the fact that the spectral projectors depend continuously on $h \in \mathcal{H}$ (see Remark 2.1), we obtain that

$$G(u_k) \rightarrow G_h(u) = 0,$$

where G_h is the defining operator of the manifold \mathcal{N}_h .

By Lemma 5.5, 0 is in the domain of $G_{u_k}^{-1}$ provided k is large enough. Setting $\tilde{u}_k = G_{u_k}^{-1}(0)$, we have that $\tilde{u}_k \in \mathcal{N}$ and $\|u_k - \tilde{u}_k\| \rightarrow 0$. As consequence, $J(\tilde{u}_k) \rightarrow J_h(u)$. This implies immediately that $m \leq m_h + \varepsilon$. Since ε is an arbitrary positive number, $m \leq m_h$.

Interchanging the role of J and J_h in the previous argument, we see that $m_h \leq m$, and the proof is complete. \square

6 Palais-Smale Sequences

Remind that a *Palais-Smale sequence* for the functional J at level c is a sequence $u_n \in H^1(\mathbb{R})$ such that $J(u_n) \rightarrow c$ and $J'(u_n) \rightarrow 0$ strongly in $H^{-1}(\mathbb{R})$ (equivalently, $\nabla J(u_n) \rightarrow 0$ strongly in $H^1(\mathbb{R})$). Also we consider Palais-Smale sequences for the restriction $J|_{\mathcal{N}}$ of the functional J to the generalized Nehari manifold \mathcal{N} . These are defined similarly. Namely, a sequence $u_n \in \mathcal{N}$ is a Palais-Smale sequence for $J|_{\mathcal{N}}$ at level c if $J(u_n) \rightarrow c$ and $\nabla_{\tau} J(u_n) \rightarrow 0$ strongly in $H^1(\mathbb{R})$, where ∇_{τ} stands for the tangent component of the gradient.

Proposition 6.1 *Let $u_n \in H^1(\mathbb{R})$ be a Palais-Smale sequence for J at level c . Then the sequence u_n is bounded in $H^1(\mathbb{R})$. Furthermore, $u_n \rightarrow 0$ strongly in $H^1(\mathbb{R})$ if and only if $c = 0$.*

Proof. Since $J(u_n)$ is bounded and $\|J'(u_n)\|_* \rightarrow 0$, we have that

$$I(u_n) \leq C + \varepsilon_n \|u_n\|,$$

where $\varepsilon_n \rightarrow 0$. Then inequality (5.2) of Lemma 5.2 yields

$$\|u_n\|^2 \leq C(\|u_n\| + \varepsilon_n^{1/2} \|u_n\|^{3/2} + \varepsilon_n \|u_n\|^2).$$

This implies the boundedness of u_n .

If $c = 0$, then the boundedness of u_n and inequality (5.2) imply that $\|u_n\| \rightarrow 0$. The converse implication is trivial. □

Proposition 6.2 *Every Palais-Smale sequence for $J|_{\mathcal{N}}$ is a Palais-Smale sequence for J .*

Proof. Let $u_n \in \mathcal{N}$ be a Palais-Smale sequence for $J|_{\mathcal{N}}$. Inequality (5.2) of Lemma 5.2 implies immediately that the sequence u_n is bounded. Let $g_n = \nabla J(u_n)$ and g_n^{τ} be the tangent component of g_n , i.e., the orthogonal projection of g_n onto the tangent space to \mathcal{N} at u_n . Then, by assumption, $g_n^{\tau} \rightarrow 0$. We have to show that, actually, $g_n \rightarrow 0$.

Since the sequence u_n is bounded, then, by Lemma 5.4(b), $\|P_n\| \leq C$ for some $C > 0$, where P_n is the projector onto \bar{E}_{u_n} along the tangent space T_{u_n} . The adjoint operator, P_n^* , is the projector onto the orthogonal complement to \bar{E}_{u_n} along the normal subspace to \mathcal{N} at u_n , and $\|P_n^*\| \leq C$ for some $C > 0$ independent of n .

Now notice that, by the definition of \mathcal{N} , g_n is orthogonal to the subspace \bar{E}_{u_n} . Therefore, $g_n = P_n^* g_n^{\tau}$ and, hence, $\|g_n\| \leq C \|g_n^{\tau}\| \rightarrow 0$. This completes the proof. □

Proposition 6.3 *If u_n is a Palais-Smale sequence for J at a level $c > 0$, then there exists a Palais-Smale sequence $\tilde{u}_n \in \mathcal{N}$ at the same level such that $\|u_n - \tilde{u}_n\| \rightarrow 0$, and $c \geq m$.*

Proof. By Proposition 6.1, the sequence u_n is bounded. Hence,

$$J(u_n) - \frac{1}{2}(J'(u_n), u_n) = \frac{1}{2} \int_{\mathbb{R}} f(x, u_n) u_n dx - \int_{\mathbb{R}} F(x, u_n) dx \rightarrow c.$$

Since $F(x, u) \geq 0$, we obtain that

$$\int_{\mathbb{R}} f(x, u_n) u_n dx \geq c$$

for all n large enough. Furthermore, Palais-Smale property also implies that $G(u_n) \rightarrow 0$. By Lemma 5.5,

$$\tilde{u}_n = G_{u_n}^{-1}(0) \in \mathcal{N}$$

is well-defined for all n large enough, and $\|u_n - \tilde{u}_n\| \rightarrow 0$. Obviously, \tilde{u}_n is a Palais-Smale sequence at the level c . Since $J(\tilde{u}_n) \geq m$, the last statement of proposition follows immediately. \square

Combining Propositions 5.1 and 6.3, we obtain

Corollary 6.1 *If u_n is a Palais-Smale sequence for J at a positive level, then $\liminf \|u_n\|_{\infty} \geq \varepsilon_0 > 0$, where ε_0 is the constant from Proposition 5.1.*

The following proposition is one of our key ingredients.

Proposition 6.4 *Given $\varepsilon > 0$, there exists a Palais-Smale sequence u_n for the functional $J|_{\mathcal{N}}$ (hence, for J) at some level $c \in [m, m + \varepsilon]$ such that*

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0 \tag{6.1}$$

i.e., a (\overline{PS}) sequence.

Proof. On the manifold \mathcal{N} we consider the following initial-value problem

$$\frac{d\zeta}{dt} = -\nabla_{\tau} J(\zeta), \quad \zeta(0) = u_0 \in \mathcal{N}. \tag{6.2}$$

The right-hand side of the differential equation in (6.2) is locally bounded and Lipschitz continuous. Since \mathcal{N} is a $C^{1,1}$ manifold, we see that problem (6.2) has a unique local solution $\zeta(t) \in \mathcal{N}$ for any $u_0 \in \mathcal{N}$. Indeed, the problem reduces to an initial-value problem on a ball in T_{u_0} centered at 0, with Lipschitz continuous right-hand side. Moreover, if $\|u_0\| \leq R$ for some $R > 0$, then, by Proposition 5.3, both the radius of the ball on which the reduced problem is defined and the Lipschitz constant of the right-hand side depend only on R , not on u_0 . This implies that the local solution is defined on a time interval whose length is bounded below by a positive constant that depends only on R .

If $\zeta(t)$ is a solution of (2.4), then $J(\zeta(t))$ is a non-increasing function of t . Hence, $J(\zeta(t)) \leq J(u_0)$ for all positive t in the domain of the solution.

Therefore, by inequality (5.2) of Lemma 5.2, $\|\zeta(t)\| \leq R$ on the domain of ζ , where $R > 0$ depends only on $J(u_0)$. This implies that the solution is defined for all $t > 0$.

Now we choose any $u_0 \in \mathcal{N}$ such that $J(u_0) \leq m + \varepsilon$. Then

$$J(\zeta(t)) \rightarrow c \in [m, m + \varepsilon]$$

as $t \rightarrow \infty$. Let

$$\varphi(t) = - \int_0^t \|\nabla_\tau J(\zeta(s))\|^2 ds.$$

Then

$$\varphi(t) = J(\zeta(t)) - J(u_0)$$

and

$$\lim_{t \rightarrow \infty} \varphi(t) = \inf_{t > 0} \varphi(t) \geq -\varepsilon.$$

Let $s_n \rightarrow \infty$ be a sequence such that $|s_n - s_{n-1}| \rightarrow 0$. Since s_n is a minimizing sequence for φ , Ekeland's variational principle implies the existence of a sequence t_n such that $\varphi(t_n) \rightarrow \inf_{t > 0} \varphi(t)$, $\varphi'(t_n) \rightarrow 0$ and $t_n - s_n \rightarrow 0$. Setting $u_n = \zeta(t_n)$, we obtain a Palais-Smale sequence for $J|_{\mathcal{N}}$. (Alternatively, at this point one can use an elementary argument from Real Analysis instead of Ekeland's principle). Finally, since $\zeta(t)$ is bounded, $\nabla_\tau(\zeta(t))$ is bounded as well. Hence, by the mean value theorem and the equation for ζ ,

$$\|u_n - u_{n-1}\| \leq C|t_n - t_{n-1}| \rightarrow 0.$$

This completes the proof. \square

In what follows we consider Palais-Smale sequences at levels close to m . The next result shows that the structure of such sequences is much simpler than in general case.

Lemma 6.1 *Let u_n be a Palais-Smale sequence for J at a level $c \in [m, 2m)$. Suppose that $u_n \rightarrow u_0$ weakly in $H^1(\mathbb{R})$.*

(a) If $u_0 \neq 0$, then $u_n \rightarrow u_0$ strongly in $H^1(\mathbb{R})$, u_0 is a critical point of J , and $J(u_0) = c$.

(b) If $u_0 = 0$, then there exist a sequence $x_n \in \mathbb{R}$, with $\lim |x_n| = \infty$, and a nontrivial critical point v_h of J_h for some $h \in \mathcal{H}$, with $J_h(v_h) = c$, such that along a subsequence $T_{x_n} u_n \rightarrow v_h$ and $u_n - T_{x_n} v_h \rightarrow 0$ strongly in $H^1(\mathbb{R})$.

Proof. (a) Since J' is weakly continuous, $J'(u_0) = 0$. By Proposition 4.2, $u_n - u_0$ is a Palais-Smale sequence at level $c - J(u_0)$. If $c - J(u_0) > 0$, then, by Proposition 6.3, $c - J(u_0) \geq m$ which is impossible because $c < 2m$ while $J(u_0) \geq m$. Thus, $J(u_0) = c$ and $u_n - u_0$ is a Palais-Smale sequence at level zero. By Proposition 6.1, $u_n - u_0 \rightarrow 0$ strongly in $H^1(\mathbb{R})$.

(b) Let $x_n \in \mathbb{R}$ be any point of global maximum for the function $|u_n|$ (obviously, such points exist), and let $v_n = T_{x_n} u_n$. Since $u_n \rightarrow 0$ weakly, Corollary 6.1 implies that $|x_n| \rightarrow \infty$. Furthermore, zero is not a weak limit point of

the sequence v_n . Since v_n is a bounded sequence, then, along a subsequence, $v_n \rightarrow v_h \neq 0$ weakly in $H^1(\mathbb{R})$. Passing to a subsequence one more time, we also obtain limit potential V_h and nonlinearity f_h , and, hence, the limit functional J_h . By Proposition 4.1 and weak continuity of J'_h , we obtain easily that $J'_h(v_h) = 0$.

By Propositions 4.1 and 4.2,

$$\begin{aligned} J(u_n - T_{-x_n}v_h) - J(u_n) + J_h(v_h) &= \\ &= (J_{x_n}(v_n - v_1) - J_{x_n}(v_n) + J_{x_n}(v_h)) - \\ &\quad - (J_{x_n}(v_h) - J_h(v_h)) \rightarrow 0. \end{aligned}$$

Hence, $J(u_n - T_{-x_n}v_h) \rightarrow c - J_h(v_h)$. Similarly, making use of second parts of Propositions 4.1 and 4.2 we see that $J'_h(u_n - T_{-x_n}v_h) \rightarrow 0$ in $H^1(\mathbb{R})$ and, hence, $u_n - T_{-x_n}v_h$ is a Palais-Smale sequence for J at the level $c - J_h(v_h)$. As in the proof of first part of Proposition, we see that $J_h(v_h) = c$ and, by Proposition 6.1,

$$\|u_n - T_{-x_n}v_h\| = \|v_n - v_h\| \rightarrow 0.$$

This completes the proof. □

7 Proof of Main Result

Theorem 3.1 is an immediate consequence of the following two propositions.

Proposition 7.1 *If $u \in H^1(\mathbb{R})$ is a nontrivial solution of equation (3.1), then u' is a continuous function and*

$$0 < |u(x)| + |u'(x)| \leq \alpha \exp(-\beta|x|)$$

for some positive constants α and β .

Proof. Let $u \in H^1(\mathbb{R})$ be a nonzero solution. Set $V_1(x) = f(x, u(x))/u(x)$ for all x such that $u(x) \neq 0$ and $V_1(x) = 0$ otherwise. Then the function u is an L^2 -eigenfunction of the operator $L - V_1(x)$ with the eigenvalue zero. It is easily seen that $V_1 \in L^\infty(\mathbb{R})$ and $V_1(x)$ vanishes at infinity in the sense that $\text{ess sup}_{|x| \geq R} V_1 \rightarrow 0$ as $R \rightarrow \infty$. Hence, $V_1(x)$ is a relatively compact perturbation of the operator L , and outside of $\sigma(L)$ the perturbed operator may have only isolated eigenvalues of finite multiplicity. Now the result follows immediately from well-known properties of eigenfunctions of Schrödinger operators (see, *e.g.*, [25]). □

Proposition 7.2 *For every $\varepsilon > 0$ there exists a critical point of the functional J with critical value $c \in [m, m + \varepsilon]$.*

Proof. Without loss of generality, we suppose that $\varepsilon < m$. Let u_n be the Palais-Smale sequence from Proposition 6.4. We consider two cases.

Case 1. The sequence u_n has a non-zero weak limit point u_0 . Then, by Lemma 6.1(a), u_0 is a critical point of J at the level c , and we obtain the required.

Case 2. The sequence u_n converges to zero weakly in $H^1(\mathbb{R})$. For $u \in H^1(\mathbb{R})$ we set

$$r(x; u) = \int_x^\infty [(u')^2(z) + u^2(z)] dz.$$

The function $r(x; u)$ is continuous, non-increasing, and $r(x; u) \rightarrow 0$ as $x \rightarrow \infty$. By Proposition 5.1, there exists $x_n \in \mathbb{R}$ such that $r(x_n, u_n) = \delta_0$, where $\delta_0 = \varepsilon_0^2/2$. Note that x_n is not necessarily unique.

We claim that $x_n - x_{n-1} \rightarrow 0$ and $|x_n| \rightarrow \infty$. Consider any subsequence $x_{n'}$ of x_n . By Lemma 6.1(b), there exists a subsequence $u_{n''}$, a sequence $y_{n''} \in \mathbb{R}$, with $\lim |y_{n''}| = \infty$, and a nontrivial critical point v_h of J_h , for some $h \in \mathcal{H}$, such that $T_{y_{n''}} u_{n''} \rightarrow v_h$ strongly in $H^1(\mathbb{R})$. Hence, $T_{y_{n''}} u_{n''-1} \rightarrow v_h$ strongly in $H^1(\mathbb{R})$. This implies that

$$r(x; T_{y_{n''}} u_{n''}) \rightarrow r(x; v_h)$$

and

$$r(x; T_{y_{n''}} u_{n''-1}) \rightarrow r(x; v_h)$$

in $L^\infty(\mathbb{R})$. By Proposition 7.1, the function $r(x; v_h)$ is strictly decreasing, and there exists a unique $x_h \in \mathbb{R}$ such that $r(x_h; v_h) = \delta_0$. Now it is easily seen that

$$\lim x_{n''} - y_{n''} = \lim x_{n''-1} - y_{n''} = x_h.$$

This implies the claim immediately.

Setting $v_n = T_{x_n} u_n$, we show that 0 is not a weak limit point of the sequence v_n . Indeed, assume the contrary. Since u_n is a Palais-Smale sequence for J , then, along a subsequence, v_n is a Palais-Smale sequence for some functional in the envelope of J , and $v_n \rightarrow 0$ weakly in $H^1(\mathbb{R})$. By Lemma 6.1(b), passing to a further subsequence, there exists a sequence y_n , yet another functional J_h and its nontrivial critical point v_h such that $|y_n| \rightarrow \infty$, $T_{y_n} v_n \rightarrow v_h$ and $T_{-y_n} v_h - v_n \rightarrow 0$ strongly in $H^1(\mathbb{R})$. This implies that

$$r(0; v_n) - r(0; T_{-y_n} v_h) = \delta_0 - r(0; T_{-y_n} v_h) \rightarrow 0.$$

On the other hand, since $|y_n| \rightarrow \infty$, we see that, along a further subsequence, either

$$r(0; T_{-y_n} v_h) \rightarrow 0,$$

or

$$r(0; T_{-y_n} v_h) \rightarrow \|v_h\|^2 \geq 2\delta_0,$$

a contradiction.

Now suppose for definiteness that the sequence x_n is unbounded above (the other case being similar). As it is well-known (see, *e.g.*, [11, 15]), there exists a returning sequence $z_k \rightarrow \infty$ for almost periodic functions $V(x)$ and $f_u(x, u)$ in the sense that

$$T_{z_k} V \rightarrow V$$

in $BS(\mathbb{R})$ and

$$\|T_{z_k} f_u(\cdot, u) - f_u(\cdot, u)\|_R \rightarrow 0 \quad \forall R > 0.$$

Since $x_n - x_{n-1} \rightarrow 0$, there exists a subsequence x_{n_k} such that $z_k - x_{n_k} \rightarrow 0$. Along a subsequence, $v_{n_k} \rightarrow v \neq 0$ weakly in $H^1(\mathbb{R})$. We shall show that v is a solution of the problem. By Proposition 4.1,

$$\begin{aligned} & \|J'(v_{n_k}) - J'_{x_{n_k}}(v_{n_k})\|_* \leq \\ & \leq \|J'(v_{n_k}) - J'_{z_k}(v_{n_k})\|_* + \|J'_{z_k}(v_{n_k}) - J'_{x_{n_k}}(v_{n_k})\|_* \rightarrow 0. \end{aligned}$$

Then the weak continuity of J' and the fact that u_n is a Palais-Smale sequence for J imply that

$$\begin{aligned} (J'(v), \varphi) &= \lim(J'(v_{n_k}), \varphi) = \\ &= \lim(J'_{x_{n_k}}(v_{n_k}), \varphi) = \lim(J'(u_{n_k}), T_{-x_{n_k}} \varphi) = 0. \end{aligned}$$

This completes the proof. □

8 An Application

Suppose that a dielectric medium occupies the whole space \mathbb{R}^3 , and its material characteristics depend on the x -variable only. Notice that in such a medium the magnetic permeability is equal to 1 and, hence, the magnetic induction is equal to the magnetic field. Considering electromagnetic fields that depend on the time t and the x -variable only, we concentrate on a special class of such fields, the so-called *TE*-modes. In a *TE*-mode the electric and magnetic components are of the form $(E, 0, 0)$ and $(0, H_1, H_2)$, respectively. Then the Maxwell equations reduce to the following equation

$$-\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 \mathcal{F}(E)}{\partial t^2}$$

for the electric field only, where $D = \mathcal{F}(E)$ is the constitutive relation between the displacement and the electric field.

In the so-called Akhmediev-Kerr model this constitutive relation is of the form

$$D = (\varepsilon(x) + g(x)\langle E^2 \rangle)E,$$

where $\varepsilon(x)$ is the dielectric function of the medium, $g(x)$ represents nonlinear susceptibility, and $\langle \cdot \rangle$ stands for the time average. In general, the nonlinear

susceptibility may attain values of any sign. If $g(x) > 0$, the medium is self-focusing, while $g(x) < 0$ means that the medium is defocusing. However, in the following we assume that $g(x)$ does not change sign. More precisely, we suppose that the functions $\varepsilon(x)$ and $g(x)$ are measurable, bounded, and Stepanov almost periodic, $g(x)$ does not change sign, and both $\varepsilon(x)$ and $|g(x)|$ are bounded below by positive constants. Thus, we are dealing with a one-dimensional almost periodic photonic crystal which is either totally self-focusing, or totally defocusing, depending on the sign of g .

A gap soliton is represented by a time-harmonic wave

$$E = u(x) \cos(\omega t + \varphi_0),$$

where ω is a prohibited frequency and the wave profile $u(x)$ is a well-localized function vanishing at infinity. After this Ansatz, we obtain the following equation for the profile function

$$-\frac{d^2 u}{dx^2} - \omega^2 \varepsilon(x) u = g(x) u^3.$$

Notice that the frequency ω is prohibited if and only if 0 is not in the spectrum of the Schrödinger operator

$$L = -\frac{d^2}{dx^2} - \omega^2 \varepsilon(x).$$

Furthermore, since $\varepsilon(x) > 0$, the operator L is not positive definite, and the negative part of its spectrum is non-empty.

Thus, all the assumptions of Theorem 3.1 are satisfied, and we obtain that there exists an exponentially decaying wave profile $u \neq 0$. This shows that, in the framework of Akhmediev-Kerr model, one-dimensional almost periodic photonic crystals possess gap solitons for all prohibited frequencies.

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